PRIMER EJERCICIO DE TEORÍA CUÁNTICA DE CAMPOS
(impartido por Javier García)
a)

$COSA = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \phi_1 (\phi_1 A_{11} + \phi_2 A_{21} + \phi_3 A_{31}) +$
$\phi_2 (\phi_1 A_{12} + \phi_2 A_{22} + \phi_3 A_{32}) + \phi_3 (\phi_1 A_{13} + \phi_2 A_{23} + \phi_3 A_{33}) = \phi_1^2 A_{11} + \phi_2^2 A_{22} +$
$\phi_3^2 A_{33} + \phi_1\phi_2 A_{12} + \phi_1\phi_2 A_{21} + \phi_1\phi_3 A_{13} + \phi_1\phi_3 A_{31} + \phi_2\phi_3 A_{23} + \phi_2\phi_3 A_{32}$

Pero teniendo en cuenta que la matriz A debe ser simetrica para poder ser diagonalizada (aunque no es imprescindible para ello) $A_{ij} = A_{ji}$

$COSA = A_{11}\phi_1^2 + A_{22}\phi_2^2 + A_{33}\phi_3^2 + 2A_{12}\phi_1\phi_2 + 2A_{13}\phi_1\phi_3 + 2A_{23}\phi_2\phi_3$

$COSA = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$

igualando a la expresión dada de "COSA" obtenemos los elementos de la matriz A

$$A = \begin{pmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6 \end{pmatrix}$$

b)

Aplicando

$$Av = \lambda v \;\; ; \;\; \begin{pmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \;\; ;$$

$$\begin{pmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (-6-\lambda)x - \frac{1}{2}\sqrt{2}y \\ -\frac{1}{2}\sqrt{2}x + (-6-\lambda)y - \frac{1}{2}\sqrt{2}z \\ -\frac{1}{2}\sqrt{2}y + (-6-\lambda)z \end{pmatrix}$$
$= 0$

$$\begin{vmatrix} -6-\lambda & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6-\lambda & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6-\lambda \end{vmatrix}, \text{ determinant: } -\lambda^3 - 18\lambda^2 - 107\lambda - 210 = 0$$

$-\lambda^3 - 18\lambda^2 - 107\lambda - 210 = 0$, Solution is: $-5, -6, -7$

Cosa que ya debiamos saber por que en el apartado c) los coeficientes de los terminos ψ^2 son esos tres valores propios.

La matriz diagonalizada es entonces

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

para $\lambda = -5$

$$\left\{ \begin{array}{c} (-6+5)x - \frac{\sqrt{2}}{2}y = 0 \\ -\frac{\sqrt{2}}{2}y + (-6+5)z = 0 \end{array} \right\} \left\{ \begin{array}{c} -x - \frac{\sqrt{2}}{2}y = 0 \\ -\frac{\sqrt{2}}{2}y - z = 0 \end{array} \right\}$$

$-x - \frac{\sqrt{2}}{2}y = 0$, $y = -\sqrt{2}x$

$-\frac{\sqrt{2}}{2}(-\sqrt{2}x) - z = 0 \;\; ; \;\; x - z = 0 \;\; ; \;\; z = x$

o sea $y = -\sqrt{2}x$; $z = x$
lo que corresponde a un primer vector propio
$\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ o normalizando ya que $\sqrt{1+2+1} = 2$; $v_1 = \begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix}$
y para $\lambda = -6$

$$\left\{ \begin{array}{c} (-6+6)x - \frac{\sqrt{2}}{2}y = 0 \\ -\frac{1}{2}\sqrt{2}x + (-6+6)y - \frac{1}{2}\sqrt{2}z = 0 \end{array} \right\} \left\{ \begin{array}{c} -\frac{\sqrt{2}}{2}y = 0 \\ -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}z = 0 \end{array} \right\}$$

de donde obtenemos $y = 0$; $z = -x$
lo que corresponde a un segundo vector propio
$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ o normalizando ya que $\sqrt{1+1} = \sqrt{2}$; $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}$

para $\lambda = -7$

$$\left\{ \begin{array}{c} (-6+7)x - \frac{\sqrt{2}}{2}y = 0 \\ -\frac{1}{2}\sqrt{2}x + (-6+7)y - \frac{1}{2}\sqrt{2}z = 0 \\ -\frac{\sqrt{2}}{2}y + (-6+7)z = 0 \end{array} \right\} \left\{ \begin{array}{c} x - \frac{\sqrt{2}}{2}y = 0 \\ -\frac{\sqrt{2}}{2}y + z = 0 \end{array} \right\}$$

de donde obtenemos $y = \sqrt{2}x$; $z = x$
lo que corresponde al tercer vector propio
$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ o normalizando ya que $\sqrt{1+2+1} = 2$; $v_3 = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix}$
Por consiguiente

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

para comprobar esta solución aplicaremos $D = M^T AM$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{2} & \frac{1}{2} \end{pmatrix}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{2} & \frac{1}{2} \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2} \end{pmatrix}$$

0.1 $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$: $\begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}$ y nos ponemos muy contentos

Tendremos entonces

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\psi_1 + \frac{1}{2}\sqrt{2}\psi_2 + \frac{1}{2}\psi_3 \\ \frac{1}{2}\sqrt{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_1 \\ \frac{1}{2}\psi_1 + \frac{1}{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_2 \end{pmatrix}$$

c) Aplicamos simplemente que en la nueva base

$COSA = \lambda_1\psi_1^2 + \lambda_2\psi_2^2 + \lambda_3\psi_3^2$

por tanto

$COSA = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$

que se obtendría como

$COSA = \phi^T A\phi = \psi^T D\psi$

$COSA = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \end{pmatrix} \begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$

o haciendo la sustitución

$COSA = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$

$COSA =$

$-6(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_3 + \frac{1}{2}\sqrt{2}\psi_2)^2 -$

$6(\frac{1}{2}\sqrt{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_1)^2 -$

$6(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_2)^2 -$

$\sqrt{2}(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_3 + \frac{1}{2}\sqrt{2}\psi_2)(\frac{1}{2}\sqrt{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_1) -$

$\sqrt{2}(\frac{1}{2}\sqrt{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_1)(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_3 - \frac{1}{2}\sqrt{2}\psi_2) =$

$-\frac{6}{4}((\psi_1 + \psi_3 + \sqrt{2}\psi_2)^2 + (\sqrt{2}\psi_3 - \sqrt{2}\psi_1)^2 + (\psi_1 + \psi_3 - \sqrt{2}\psi_2)^2) -$

$-\frac{\sqrt{2}\sqrt{2}}{4}((\psi_1 + \psi_3 + \sqrt{2}\psi_2)(\psi_3 - \psi_1) - (\psi_3 - \psi_1)(\psi_1 + \psi_3 - \sqrt{2}\psi_2))$

Calculando por partes

$-\frac{6}{4}((\psi_1 + \psi_3 + \sqrt{2}\psi_2)^2 + (\sqrt{2}\psi_3 - \sqrt{2}\psi_1)^2 + (\psi_1 + \psi_3 - \sqrt{2}\psi_2)^2 = -6\psi_1^2 - 6\psi_2^2 - 6\psi_3^2$

$-\frac{\sqrt{2}\sqrt{2}}{4}((\psi_1 + \psi_3 + \sqrt{2}\psi_2)(\psi_3 - \psi_1) + (\psi_3 - \psi_1)(\psi_1 + \psi_3 - \sqrt{2}\psi_2)) = \psi_1^2 - \psi_3^2$

y reuniendo las partes

$-6\psi_1^2 - 6\psi_2^2 - 6\psi_3^2 + \psi_1^2 - \psi_3^2 = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$ y otra vez acabamos contentos